

MATHEMATICS

SUPPORT CENTRE

Title: Remainder Theorem and Factor Theorem

Target: On completion of this worksheet you should be able to use the remainder and factor theorems to find factors of polynomials.

Generally when a polynomial is divided by a linear expression there is a remainder.

e.g. $(3x^3 + 4x^2 - 5x + 3) \div (x + 2)$

$$\begin{array}{r}
 3x^2 - 2x - 1 \\
 x + 2 \overline{) 3x^3 + 4x^2 - 5x + 3} \\
 \underline{3x^3 + 6x^2} \\
 -2x^2 - 5x \\
 \underline{-2x^2 - 4x} \\
 -x + 3 \\
 \underline{-x - 2} \\
 5
 \end{array}$$

$$(3x^3 + 4x^2 - 5x + 3) = (x + 2)(3x^2 - 2x - 1) + 5$$

divisor

quotient

remainder

Any polynomial can be written in the following form:

polynomial \equiv divisor \times quotient + remainder.

In particular if the divisor is $(x - a)$ and the polynomial is $f(x)$ then

$$f(x) \equiv (x - a) \times \text{quotient} + \text{remainder.}$$

If $x = a$ then

$$f(a) = (a - a) \times \text{quotient} + \text{remainder.}$$

$$f(a) = \text{remainder}$$

This gives an easy way of finding the remainder when a polynomial is divided by $(x - a)$

Examples

1. Using previous example

$$\begin{aligned}
 f(x) &= 3x^3 + 4x^2 - 5x + 3 \\
 &= (x + 2)(3x^2 - 2x - 1) + 5
 \end{aligned}$$

Now using $x = -2$

$$\begin{aligned}
 f(-2) &= (-2 + 2)(3 \times (-2)^2 - 2 \times (-2) - 1) + 5 \\
 &= 0 \times (3 \times (-2)^2 - 2 \times (-2) - 1) + 5 \\
 &= 5
 \end{aligned}$$

i.e. the remainder.

2. Find the remainder when

$(2x^3 - 5x^2 + x - 3)$ is divided by $(x - 1)$

$$\text{Let } f(x) = 2x^3 - 5x^2 + x - 3$$

Substitute $x = 1$ since we require $(x - 1) = 0$

$$f(1) = 2 \times 1^3 - 5 \times 1^2 + 1 - 3 = -5$$

The remainder is -5

3. Find the remainder when

$(x^4 + 4x^2 + 3x - 7)$ is divided by $(x + 3)$

$$\text{Let } f(x) = x^4 + 4x^2 + 3x - 7$$

Substitute $x = -3$ since we require $(x + 3) = 0$

$$f(-3) = (-3)^4 + 4 \times (-3)^2 + 3 \times (-3) - 7 = 101$$

The remainder is 101

Exercise

Find the remainders for the following:

1. $(x^3 - 5x^2 + 6x - 4) \div (x - 2)$

2. $(4x^3 + 3x^2 + x + 2) \div (x - 1)$

3. $(2x^4 - x^3 + 3x^2 - 1) \div (x + 1)$

4. $(2x^3 - 6x - 5) \div (x + 3)$

5. $(x^3 - 4x^2 - x) \div (x - 4)$

(Answers: -4, 10, 5, -29, -4)

Example

Find the remainder when

$(3x^3 + 4x^2 - 5x - 2)$ is divided by $(x - 1)$

Let $f(x) = 3x^3 + 4x^2 - 5x - 2$ and $x = 1$

$$f(1) = 3 \times 1^3 + 4 \times 1^2 - 5 \times 1 - 2 = 0$$

The remainder is 0.

$$3x^3 + 4x^2 - 5x - 2 = (x - 1) \times \text{quotient} + 0 \\ = (x - 1) \times \text{quotient}$$

so $(x - 1)$ is a factor of $(3x^3 + 4x^2 - 5x - 2)$

We can use the remainder theorem to check for factors of a polynomial.

As before

$$f(x) = (x - a) \times \text{quotient} + \text{remainder}$$

and $f(a) = \text{remainder}$

If $(x - a)$ is a factor then the remainder is 0
ie $f(a) = 0$

This is called the factor theorem.

Examples

1. Is $(x - 3)$ a factor of $(2x^3 - 3x^2 - 8x - 3)$?

Let $f(x) = (2x^3 - 3x^2 - 8x - 3)$ and $x = 3$
as we are checking whether $(x - 3)$ is a factor.

$$f(3) = 2 \times 3^3 - 3 \times 3^2 - 8 \times 3 - 3 = 0$$

so $(x - 3)$ is a factor of $(2x^3 - 3x^2 - 8x - 3)$

2. Is $(x - 1)$ a factor of $(2x^3 - 3x^2 - 8x - 3)$?

Using $f(x)$ as above and $x = 1$

$$f(1) = 2 \times 1^3 - 3 \times 1^2 - 8 \times 1 - 3 = -12 \neq 0$$

so $(x - 1)$ is not a factor of $(2x^3 - 3x^2 - 8x - 3)$

Exercise

1. Is $(x - 1)$ a factor of

$$f(x) = (x^3 + 2x^2 - 2x - 1)?$$

2. Is $(x + 2)$ a factor of $f(x) = (4x^2 + 13x + 10)$?

3. Is $(x - 2)$ a factor of $f(x) = (4x^2 + 13x + 10)$?

4. Is $(x + 3)$ a factor of

$$f(x) = (3x^3 + 10x^2 + x - 6)?$$

5. Is $(x - 1)$ a factor of

$$f(x) = (3x^3 + 10x^2 + x - 6)?$$

(Answers: yes, yes, no, yes, no)

We can use the factor theorem to factorise polynomials, although some trial and error is involved.

Example

Factorise $(2x^3 + 5x^2 - x - 6)$.

Let $f(x) = 2x^3 + 5x^2 - x - 6$. Since the constant is -6 we will consider factors of this ie. $\pm 1, \pm 2, \pm 3, \pm 6$. We will try $(x - 1)$

$$f(1) = 2 \times 1^3 + 5 \times 1^2 - 1 - 6 = 0$$

so $(x - 1)$ is a factor.

Now we can find the quadratic factor by division or by repeating the above.

$$\begin{array}{r} 2x^2 + 7x + 6 \\ (x-1) \overline{) 2x^3 + 5x^2 - x - 6} \\ \underline{2x^3 - 2x^2} \\ 7x^2 - x \\ \underline{7x^2 - 7x} \\ 6x - 6 \\ \underline{6x - 6} \\ 0 \end{array}$$

$$f(x) = 2x^3 + 5x^2 - x - 6 \\ = (x - 1)(2x^2 + 7x + 6) \\ = (x - 1)(x + 2)(2x + 3)$$

The quadratic factor is factorised in the normal way.

Exercise

Factorise the following:

1. $f(x) = x^3 + 2x^2 - 5x - 6$

2. $f(x) = 2x^3 + x^2 - 2x - 1$

3. $f(x) = x^3 - 3x^2 - 3x - 4$

4. $f(x) = 3x^3 + 6x^2 + x + 2$

5. $f(x) = 4x^3 - 15x^2 + 17x - 6$

Answers:

1. $(x + 1)(x - 2)(x + 3)$

2. $(x + 1)(x - 1)(2x + 1)$

3. $(x - 4)(x^2 + x + 1)$

4. $(x + 2)(3x^2 + 1)$

5. $(x - 1)(x - 2)(4x - 3)$